FERMI AGE THEORY

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INTRODUCTION

We consider the description of the slowing down in energy of neutrons through interaction with the nuclei of a moderator. The analysis leads to a criticality equation that can be used to determine the critical dimensions of a medium in which neutrons are diffusing.

THE SLOWING DOWN DENSITY

We define the slowing density q to be the number of neutrons per unit volume per unit time slowing down past the energy E or its corresponding lethargy u as shown for an infinite medium in Fig. 1.

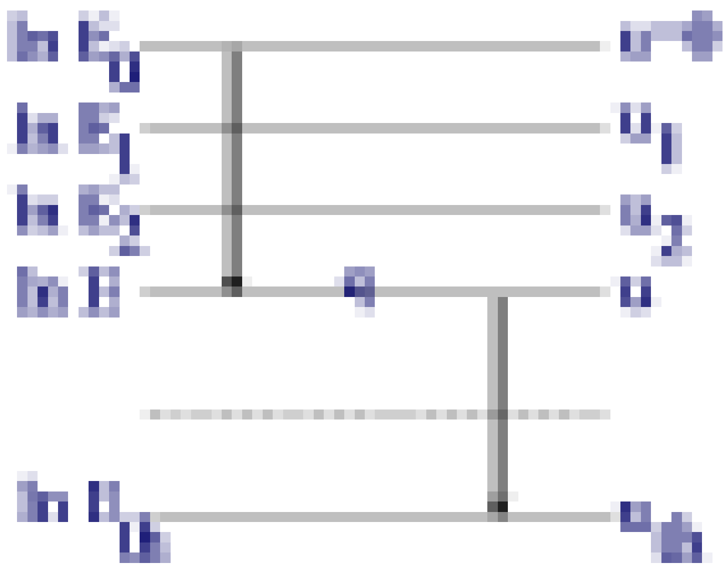

Fig. 1: Slowing down of neutrons in an infinite medium.

In this Figure, E_0 is the fission energy, and E_{th} is the thermal energy. All neutrons pass through every energy level and the slowing down density past any energy E is a constant:

$$q = q_0 \quad (1)$$

which is also the number of neutrons per unit volume per unit time produced at the fission energy E_0. This is true provided there is no absorption while slowing down, or if the absorption is weak or negligible.

THE AVERAGE LOGARITHMIC ENERGY DECREMENT

For any given scattering material of mass number A, the average logarithmic energy *loss* between energies E_1 and E_2 per collision is independent of energy and is defined as:

$$\xi = \overline{\ln E_1 - \ln E_2} = \overline{\ln \frac{E_1}{E_2}} \qquad (2)$$

It is also the *gain* in "lethargy" per collision. The lethargy, u means laziness. A gain in lethargy is a loss in energy. If E_0 is the initial energy, usually the average fission energy at 2 MeV, then its energy after slowing down to thermal energy is:

$$E_{th} = E_0 e^{-u} \qquad (3)$$

Taking the natural logarithm of both sides yields:

$$\ln \frac{E_{th}}{E_0} = -u$$

or:

$$u = \ln \frac{E_0}{E_{th}} \qquad (4)$$

For any energy E,

$$u = \ln \frac{E_0}{E} = \ln E_0 - \ln E \qquad (5)$$

and:

$$du = -\frac{dE}{E} \qquad (6)$$

where the minus sign signifies a loss of energy.

If E_{th} = 0.0323 eV, then from Eqn. 4 the neutron's lethargy is:

$$u = \ln \frac{E_0}{E_{th}} = \frac{2x10^6}{0.0323} = 17.94 \approx 18 \qquad (7)$$

If we use the variable:

$$\alpha = \left(\frac{A-1}{A+1} \right)^2 \qquad (8)$$

then the average logarithmic energy decrement is:

$$\xi = 1 + \frac{\alpha}{1-\alpha} \ln \alpha \approx \frac{2}{A + {}^2\!/_3}, \quad \forall A \geq 10 \qquad (9)$$

The n umber of collisions to thermalize a neutron from the fission energy E_0 to the thermal energy E_{th} can then be calculated from:

$$n = \frac{\ln \frac{E_0}{E_{th}}}{\xi} \qquad (10)$$

For a hydrogen moderator:

$$\xi = 1,$$

and from Eqns. 7 and 10, the number of collisions for a neutron to thermalize in a hydrogenous moderator is:

$$n \approx \frac{18}{\xi} = \frac{18}{1} = 18 \text{ collisions.}$$

In contrast in graphite or a carbon moderator, from Eqn. 9:

$$\xi = \frac{2}{12 + {}^2\!/_3} = 0.1589$$

and the number of collisions to thermalization becomes:

$$n = \frac{18}{0.1589} = 113 \text{ collisions.}$$

It becomes apparent that a hydrogenous material is a better neutron moderator than graphite.

If a reactor is made up of a mixture of several materials, one can estimate the average logarithmic energy decrement from:

$$\xi_{mixture} = \sum_{i=1}^{N} \frac{\xi_i \Sigma_{si}}{\Sigma_{si}} \tag{11}$$

where N is the number of isotopes in the mixture and Σ_{si} is the macroscopic scattering cross section for the i-th isotope in the mixture.

THE REMOVAL CROSS SECTION

We need to define the probability per unit length of a neutron losing all its energy above thermal. This is also called the fast neutrons removal cross section:

$$\Sigma_1 = \frac{\text{probability per unit length of a logarithmic energy loss}}{\text{total energy loss}}$$

Thus:

$$\Sigma_1 = \frac{\xi\Sigma_s}{\int_{E_{th}}^{E_0} \frac{dE}{E}} = \frac{\xi\Sigma_s}{\ln \frac{E_0}{E_{th}}} \tag{12}$$

THE INVERSE ENERGY 1/E-FLUX DEPENDENCE

For a lethargy interval Δu, $\frac{\Delta u}{\xi}$ is the probability of a neutron making a collision in the interval Δu, since ξ is the lethargy gain per collision. Since q is the number of neutrons per unit volume per unit time passing through Δu, $q\frac{\Delta u}{\xi}$ is the number of collisions per unit volume per unit time in the interval Δu. Replacing Δu by du, and since:

$$du = -\frac{dE}{E},$$

we can write:

$$\frac{\text{Number of collisions in interval dE}}{\text{unit volume . unit time}} = q\frac{du}{\xi} = -\frac{q}{\xi E}dE$$

Since absorptions are negligible, this is also equal to $-\Sigma_s \phi dE$, where ϕ is the neutron flux. Thus:

$$\Sigma_s \phi dE = \frac{q}{\xi E} dE \ ,$$

or:

$$\phi = \frac{q}{\xi \Sigma_s} \frac{1}{E} \qquad (13)$$

For a weakly absorbing medium, the slowing down density q = q_0, a constant. If the scattering cross-section is also constant, then the flux is inversely proportional to the neutron energy:

$$\phi \propto_{\ell} \frac{1}{E} \qquad (14)$$

THE FERMI AGE EQUATION

For a weakly absorbing medium of finite size, a neutron balance equation can be written for steady state and an interval of energy dE as:

$$\begin{aligned} &\left[\text{Neutrons produced in interval dE}\right] \\ &-\left[\text{Neutrons absorbed in dE}\right] \\ &-\left[\text{Neutrons leaking out of dE}\right] = 0 \end{aligned} \qquad (15)$$

This can be expressed mathematically as:

$$SdE - \Sigma_a \phi dE + D\nabla^2 \phi dE = 0 \qquad (16)$$

The source term for the interval dE is the number of neutrons slowing into dE minus the number of neutrons slowing out of dE:

$$SdE = q(E + dE) - q(E) = \frac{\partial q}{\partial E} dE \qquad (17)$$

If we can neglect the absorptions in the fast region, then $\Sigma_a \simeq 0$, and we can write Eqn. 16 as:

$$D(E)\nabla^2 \phi(E,r) dE = -\frac{\partial q(E,r)}{\partial E} dE \qquad (18)$$

where we are now showing the dependence on the spatial position r, and the energy E.

If we rewrite Eqn. 13 as:

$$\phi(E,r) = \frac{1}{\xi\Sigma_s}\frac{q(E,r)}{E} \tag{13)'$$

then Eqn. 18 can be written as:

$$\frac{D(E)}{\xi\Sigma_s E}\nabla^2 q(E,r) = -\frac{\partial q(E,r)}{\partial E} \tag{14}$$

Placing the spatial derivative at the left hand side, and the energy derivative at the right hand side, we get:

$$\nabla^2 q(E,r) = -\frac{\xi\Sigma_s E}{D(E)}\frac{\partial q(E,r)}{\partial E} \tag{15}$$

Let us define the quantity:

$$d\tau = -\frac{D}{\xi\Sigma_s}\frac{dE}{E} \tag{16}$$

where τ is called the Fermi Age, then we can write the "Fermi Age Equation" as:

$$\nabla^2 q = \frac{\partial q}{\partial \tau} \tag{17}$$

Notice that the Fermi Age τ has units of area, and not time. However Eqn. 17 has the form of the parabolic Diffusion Equation encountered in mathematical physics where normally τ is the time, hence the name given to it.

The Fermi Age of thermal neutrons can be estimated from Eqn. 16 as:

$$\tau = \int d\tau = -\int_{E_0}^{E_{th}} \frac{D}{\xi\Sigma_s}\frac{dE}{E} = +\frac{\bar{D}_F}{\xi\bar{\Sigma}_s}\ln\frac{E_0}{E_{th}} \tag{18}$$

where $\bar{D}_F$ is the value of the diffusion coefficient D and of the macroscopic scattering cross section Σ_s averaged over the energy range.

From Eqn. 12, we can write an expression of the Fermi Age in terms of the fast neutron removal cross section:

$$\tau = \frac{\bar{D}_F}{\Sigma_1} \tag{19}$$

which appears analogous to the expression for the thermal diffusion area:

$$L_{th}^2 = \frac{D_{th}}{\Sigma_{ath}} \tag{20}$$

If Σ_s is independent of energy, and fast absorptions are negligible, the diffusion coefficient can be estimated from:

$$\bar{D}_F = \frac{1}{3\Sigma_s(1-\bar{\mu}_0)} \tag{21}$$

If on the other hand, experimental values of the age τ are known, then $\bar{D}_F$ can be determined instead from Eqn. 19.

REACTOR CRITICALITY EQUATION BASED ON AGE THEORY

The thermal neutron diffusion equation from one-group theory can be written as:

$$D\nabla^2\phi - \Sigma_a\phi + k_\infty \ell_f \Sigma_a\phi = 0 \tag{22}$$

where ℓ_f is the neutrons fast non-leakage probability.

Without fast absorptions, we say that: $q(E) = q_0$. If absorptions exist, we define the resonance escape probability p as:

$$p(E) = \frac{q(E)}{q_0} \tag{23}$$

or:

$$q(\tau) = pq(\tau = 0) \tag{24}$$

Since the source term in the diffusion equation can be written as:

$$
\begin{aligned}
S &= k_\infty \ell_f \Sigma_a \phi \\
&= \eta \varepsilon p f \ell_f \Sigma_a \phi \\
&= \frac{\nu \Sigma_f}{\Sigma_{a,fuel}} \varepsilon p \frac{\Sigma_{a,fuel}}{\Sigma_a} \ell_f \Sigma_a \phi \\
&= \nu \Sigma_f \varepsilon p \ell_f \phi
\end{aligned} \tag{25}
$$

where we used the definition of the infinite medium multiplication factor as the four-factor formula:

$$
k_\infty = \eta \varepsilon p f \ , \tag{26}
$$

the definition of the regeneration factor as:

$$
\eta = \frac{\nu \Sigma_f}{\Sigma_{a,fuel}} \tag{26}
$$

and the definition of the fuel utilization factor as:

$$
f = \frac{\Sigma_{a,fuel}}{\Sigma_a} \tag{27}
$$

By comparing Eqns. 24 and 25, we can deduce that:

$$
\begin{aligned}
S &= \nu \Sigma_f \varepsilon p \ell_f \phi \\
&= p \ell_f (\nu \Sigma_f \varepsilon \phi) \\
&= p \ell_f q(\tau = 0)
\end{aligned} \tag{28}
$$

where:

$$
q(\tau = 0) = \nu \Sigma_f \varepsilon \phi \tag{29}
$$

Returning to the Age Equation, let us separate the variables as:

$$
q(\tau, r) = F(\tau) R(r) \tag{30}
$$

and substitute in Eqn. 17 to get:

$$\begin{aligned}
\nabla^2 q(\tau, r) &= \frac{\partial q(\tau, r)}{\partial \tau} \\
\nabla^2 [F(\tau)R(r)] &= \frac{\partial [F(\tau)R(r)]}{\partial \tau} \\
F(\tau)\nabla^2 R(r) &= R(r)\frac{\partial F(\tau)}{\partial \tau} \\
\frac{\nabla^2 R(r)}{R(r)} &= \frac{1}{F(\tau)}\frac{\partial F(\tau)}{\partial \tau}
\end{aligned} \tag{31}$$

For a large unreflected or bare reactor, we can assume that the slowing down density has the same distribution as the thermal flux:

$$\frac{\nabla^2 R(r)}{R(r)} = \frac{\nabla^2 \phi(r)}{\phi(r)} = -B_g^2 \tag{32}$$

where B_g^2 is the geometric buckling.
Thus:

$$\frac{1}{F(\tau)}\frac{dF(\tau)}{d\tau} = -B_g^2 \tag{33}$$

Notice that the partial derivative is here replaced by a total derivative.

Separating the variables and integrating:

$$\int \frac{dF(\tau)}{F(\tau)} = -B_g^2 \int d\tau$$

which has a solution:

$$\begin{aligned}
\ln F(\tau) - \ln C &= -B_g^2 \tau \\
\ln \frac{F(\tau)}{C} &= -B_g^2 \tau
\end{aligned}$$

Taking the exponential of both sides:

$$F(\tau) = Ce^{-B_g^2 \tau} \tag{34}$$

Using Eqn. 30 we get:

(35)

For an initial condition at $\tau = 0$ we can use Eqn. 29 to estimate the value of the constant C:

$$q(0,r) = CR(r) = \nu\varepsilon\Sigma_f \phi$$

from which:

$$C = \frac{\nu\varepsilon\Sigma_f \phi}{R(r)}.$$

Substituting in Eqn. 34:

$$q(\tau) = \nu\varepsilon\Sigma_f \phi e^{-B_g^2 \tau} \tag{36}$$

The number of neutrons slowing down to the thermal energy is then obtained by multiplying by the resonance escape probability:

$$q_{th} = pq(\tau_{th}) = p\nu\varepsilon\Sigma_f \phi e^{-B_g^2 \tau_{th}} \tag{37}$$

But $q_{th} = S$, the thermal neutrons source term in the diffusion equation, which shows that the fast nonleakage probability is:

$$\ell_f = e^{-B_{th}^2 \tau_{th}} \tag{38}$$

We can write the diffusion equation in terms of Fermi Age theory from Eqn. 22 as:

$$D\nabla^2 \phi - \Sigma_a \phi + k_\infty e^{-B_g^2 \tau_{th}} \Sigma_a \phi = 0 \tag{39}$$

Dividing by D, replacing:

$$\frac{D}{\Sigma_a} = L_{th}^2$$

$$\nabla^2 \phi - \frac{\Sigma_a}{D} \phi + k_\infty e^{-B_g^2 \tau_{th}} \frac{\Sigma_a}{D} \phi = 0$$

$$\nabla^2 \phi + \frac{(k_\infty e^{-B_g^2 \tau_h} - 1)}{L_{th}^2} \phi = 0$$

Dividing by ϕ and substituting:

$$\frac{\nabla^2 \phi}{\phi} = -B_g^2$$

we get:

$$\frac{k_\infty e^{-B_g^2 \tau_{th}} - 1}{L_{th}^2} = B_g^2 \qquad (40)$$

Rearranging, we get the Fermi Age criticality equation as:

$$k_{eff} = 1 = \frac{k_\infty e^{-B_g^2 \tau_{th}}}{1 + L_{th}^2 B_g^2} \qquad (41)$$

SPECIAL CASES OF THE FERMI AGE THEORY CRITICALITY EQUATION

If the value of $B_g^2 \tau_{th}$ is small, we can expand the exponential as:

$$e^{-B_g^2 \tau_{th}} \approx 1 - B_g^2 \tau_{th} \approx \frac{1}{1 + B_g^2 \tau_{th}} \qquad (42)$$

which is analogous to the expression for the fast nonleakage probability encountered in the one-group theory diffusion theory:

$$\ell_f = \frac{1}{1 + B_g^2 L_f^2} \qquad (43)$$

If the value of L_f^2 is needed in one or two-group diffusion theory, we solve the criticality Eqn. 41 for the buckling:

$$1 + L_{th}^2 B_g^2 = k_\infty e^{-B_g^2 \tau_{th}} ,$$

by iteration or by trial and error. A good first approximation is as given by Eqn. 42, which leads to:

$$(1 + L_{th}^2 B_g^2)(1 + \tau B_g^2) \simeq k_\infty$$
$$1 + (L_{th}^2 + \tau) B_g^2 + L_{th}^2 \tau B_g^4 \simeq k_\infty$$

For a large reactor the fourth power of the buckling can be considered as small relative to the second power of the buckling, consequently:

$$1 + (L_{th}^2 + \tau)B_g^2 \simeq k_\infty$$
$$1 + M^2 B_g^2 \simeq k_\infty$$

where the “migration area” is defined as:

$$M^2 = (L_{th}^2 + \tau) \tag{44}$$

This yields the modified one group diffusion theory criticality equation for a large reactor:

$$k_{eff} = 1 = \frac{k_\infty}{1 + M^2 B_g^2} \tag{45}$$

Knowing B_g^2, one can calculate L_f^2 from Eqn. 43 as:

$$\ell_f = \frac{1}{1 + B_g^2 L_f^2} = e^{-B_g^2 \tau_{th}}$$

or:

$$L_f^2 = \frac{e^{-B_g^2 \tau_{th}} - 1}{B_g^2} \tag{46}$$

CRITICALITY CALCULATION OF A MODERATED HOMOGENEOUS REACTOR BASED ON AGE THEORY

As an exam ple of the use of Fermi Age theory for criticality calculations, we consider a graphite moderated reactor core with a migration area:

$$M^2 = 3040\ [cm^2],$$

carbon moderator to uranium atomic ratio of:

$$\frac{N_C}{N_U} = 10^4,$$

regeneration factor:

$$\eta = 2.08,$$

fuel utilization factor:

$$f = 0.988,$$

resonance escape probability:

$$p \simeq 10 \text{ ,}$$

fast fission factor:

$$\varepsilon \simeq 1 \quad ,$$

and age in graphiteas:

$$\tau_{th}(graphite) \simeq 350[cm^2].$$

From Eqn. 26 the infinite medium multiplication factor is:

$$\begin{aligned} k_\infty &= \eta\varepsilon pf \\ &= 2.08x1x1x0.988 \\ &= 2.06 \end{aligned}$$

Using Eqn. 41 for criticality in Fermi Age theory:

$$\begin{aligned} 1 &= \frac{k_\infty e^{-B_g^2\tau_{th}}}{1+L_{th}^2B_g^2} \\ &= \frac{k_\infty e^{-B_g^2\tau_{th}}}{1+(M^2-\tau_{th})B_g^2} \\ &= \frac{2.06e^{-350B_g^2}}{1+(3040-350)B_g^2} \end{aligned}$$

Rearranging we get:

$$1+2690B_g^2 = 2.06e^{-350B_g^2}$$

We can deduce the following transcendental equation to be solved graphically or by iteration for the value of the critical buckling:

$$B_{g,n+1}^2 = \frac{2.06e^{-350B_{g,n}^2} - 1}{2690} \tag{47}$$

A best starting value for the buckling is one that can be estimated from the modified one group diffusion theory as:

$$B_1^2 = 3.4900x10^{-4}$$

Applying the iteration formula in Eqn. 47, we get:

$$B_2^2 = 3.0600x10^{-4}$$
$$B_3^2 = 3.1627x10^{-4}$$
$$B_4^2 = 3.1380x10^{-4}$$
$$B_5^2 = 3.1440x10^{-4}$$
$$B_6^2 = 3.1426x10^{-4}$$
$$B_7^2 = 3.1429x10^{-4}$$
$$B_8^2 = 3.1428x10^{-4}$$
$$B_9^2 = 3.1428x10^{-4}$$

Thus the solution converged at the ninth iteration to four decimal places.

For a spherical reactor core the critical radius would be:

$$R_c = \left(\frac{\pi^2}{3.148x10^{-4}}\right)^{1/2} = 177.21[cm] \quad (168.17)$$

For a cube, the critical side length would be;

$$a_c = \left(\frac{3\pi^2}{3.148x10^{-4}}\right)^{1/2} = 306.94[cm] \quad (291.28)$$

For a cylinder with a height to diameter ratio of unity:

$$H = 2R,$$

the critical radius would be:

$$R_c = \left(\frac{8.25}{3.148x10^{-4}}\right)^{1/2} = 162.02[cm] \quad (153.75)$$
$$H_c = 2R_c = 324.04[cm] \quad (307.50)$$

For comparison, the values obtained from the modified one-group theory are shown in parentheses next to each computed value. From these critical dimensions, critical volumes and critical masses can be estimated.

REFERENCES

1. M. Ragheb, "Lecture Notes on Fission Reactors Design Theory," FSL-33, University of Illinois, 1982.
2. J. R. Lamarsh, "Introduction to Nuclear Engineering," Addison-Wesley Publishing Company, 1983.