TREATMENT OF HETEROGENEOUS REACTOR SYSTEMS

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INTRODUCTION

Most reactor systems nowadays are of the heterogeneous type, where the fuel and moderator/coolant are lumped and do not form a homogeneous mixture. The criticality parameters of interest such as the regeneration factor (η) fuel utilization factor (f) and the resonance escape probability (p) used in the four factor formula for the infinite medium multiplication factor will have to account in their calculation of this fact.

CROSS-SECTIONS HOMOGENIZATION IN HETEROGENEOUS SYSTEMS

We can mix the different material regions in a heterogeneous system and treat it as a homogeneous system only if the diffusion length (L) for the homogenized region is larger than any characteristic dimension (d) of the system, for thermal neutrons:

$$L \gg d$$

Similarly for fast neutrons, the following condition must be satisfied:

$$\sqrt{\tau} \gg d \, ,$$

where: τ is the slowing down area, or age.

In any homogenization, we require the preservation of reaction rates in the heterogeneous and homogenized systems. This can be expressed as:

$$\overline{\Sigma}\,\overline{\phi}\,V = \int_V \Sigma(r)\phi(r)\,dV \tag{1}$$

where: V is the total volume of the reactor,

$\overline{\phi}$ is the average flux in the system: $\overline{\phi} = \int_V dV\,\phi(r)$

$\overline{\Sigma}$ is a homogenized cross section.

For several m separate regions, Eqn. 1 can be written as:

$$\overline{\Sigma} = \frac{\sum\limits_{i=1}^{m}(\Sigma_i\,\overline{\phi}_i\,V_i)}{\overline{\phi}\,V} = \sum\limits_{i=1}^{m}(\Sigma_i\,\frac{\overline{\phi}_i}{\overline{\phi}}\,\frac{V_i}{V}) \tag{2}$$

which is a requirement that the cross sections are to be flux weighted and volume weighted in obtaining homogeneous cross sections. If the fluxes are reasonably constant, $\overline{\phi}_i \approx \overline{\phi}$ and we can use only volume weighting:

$$\overline{\Sigma} = \frac{\sum\limits_{i=1}^{m}(\Sigma_i\,V_i)}{V} = \sum\limits_{i=1}^{m}(\Sigma_i\,\frac{V_i}{V}) \tag{3}$$

The factor $(\overline{\phi}_i / \overline{\phi})$ is often referred to as the self-shielding factor.

EFFECT OF HETEROGENEITY ON THE REGENERATION FACTOR

The quantity η is the number of fission neutron produced per neutron absorbed in the fuel. In a homogeneous reactor, "fuel" means the fissionable isotope. In a lumped

reactor, "fuel" now refers to the fissionable isotope and any isotopes which are mixed with the fissionable isotope. If the fuel is lumped, it may consist of number of isotopes, some of which are fissionable, and some are not. The average flux may not be the same in all isotopes considered as part of the fuel. In this case, the regeneration factor is given by:

$$\eta = \frac{\sum_i v_i N_i \sigma_{fi} \overline{\phi_i}}{\sum_i N_i \sigma_{ai} \overline{\phi_i}} \qquad (4)$$

where $\overline{\phi_i}$ is the average flux over the i-th isotope.

If the various isotopes are intimately mixed, $\overline{\phi_i} \approx$ constant, and Eqn. 4 becomes:

$$\eta = \frac{\sum_i v_i N_i \sigma_{fi}}{\sum_i N_i \sigma_{ai}} \qquad (5)$$

As an example, for natural uranium:

$$\eta = \frac{vN(U^{235})\sigma_f(U^{235})}{N(U^{235})\sigma_f(U^{235}) + N(U^{235})\sigma_c(U^{235}) + N(U^{238})\sigma_c(U^{238})}$$

$$= \frac{v\sigma_f(U^{235})}{\sigma_f(U^{235}) + \sigma_c(U^{235}) + \dfrac{\sigma_c(U^{238})}{r}}$$

where: $\qquad r = \dfrac{N(U^{235})}{N(U^{238})} = 0.00717 \approx \dfrac{1}{140}$.

Since cross section data bases for use in thermal reactors list the 2200 [m/sec] values, these have to be modified by the non-1/v factors (g), and the temperature and Maxwell distribution factors:

$$\sigma = \left(\frac{293}{273+t}\right)^{1/2} g\, \sigma_{2200}\, \frac{\sqrt{\pi}}{2} \tag{6}$$

where: $\dfrac{v_p}{\bar{v}} = \dfrac{\sqrt{\pi}}{2} = \dfrac{\sqrt{\dfrac{2kT}{m}}}{\sqrt{\dfrac{8kT}{\pi m}}}$ is the ratio of the most probable to the average energy

for a velocity Maxwellian distribution:

$$n(v) = 4\pi n_0 \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}}$$

g is the non-1/v factor, assuming well-moderated systems. It is a function of temperature. (Note that scattering cross sections are not subject to this correction.)

t is the medium temperature in degrees Celcius ($\approx 20^{\circ}C$ at room temperature).

Since $g(^{238}U)=1.002$, and $g(^{235}U)=0.975$, at room temperature, we get:

$$\eta = 2.47 \times \frac{0.975 \times 580}{0.975 \times 687 + \dfrac{(1.002 \times 2.75)}{0.00717}} = 1.33$$

The cross section data were taken from Table I, the non-1/v factors from Table II and the value of the average number of neutrons from fission ν from Table III.

Table I. Thermal Neutron Cross Sections
(20°C or 2200 cm/sec)

Material	Density $\rho\,[g/cm^3]$	σ_a	σ_s	σ_f
H	—	0.330	38.0	—
D	—	0.460×10^{-3}	7.0	—
Li	0.534	71.000	1.4	—
Be	1.850	10.000×10^{-3}	7.0	—
B	—	755.000	4.0	—
C	1.600	3.200	4.8	—
N	—	1.880	10.0	—
O	—	$<0.200 \times 10^{-3}$	4.2	—
Na	0.971	0.505	4.0	—
Al	2.700	0.230	1.4	—
K	0.870	1.970	1.5	—
Fe	7.860	2.530	11.0	—
Co	8.900	37.800	7.0	—
Ni	8.900	4.600	17.5	—
Mo	10.200	2.500	7.0	—
Cd	8.650	2550.000	7.0	—
In	7.280	190.000	2.2	—
Xe^{135}	—	2.700×10^6	—	—
Sm^{149}	7.700	5.000×10^4	—	—
Th^{232}	11.300	7.000	12.6	—
U^{233}	18.700	585.000	—	533
U^{235}	18.700	687.000	10.0	580
U^{238}	18.700	2.750	8.2	—
Pu^{239}	—	1065.000	9.6	750
Pu^{241}	—	—	—	1100
H_2O	1.000	0.660	108.0	—
D_2O	1.100	0.930×10^{-3}	14.3	—
BeO	3.010	9.000×10^{-3}	10.0	—

Table II. Non-1/v Factors for Absorption Cross-Sections in a Maxwellian Distribution

Element	σ_{2200}	Temperature (°C)				
		20	100	200	400	700
^{233}U	$\sigma_a = 581.37$	1.008	1.007	1.009	1.018	1.037
	$\sigma_f = 526.91$	1.001	1.012	1.014	1.022	1.036
^{235}U	$\sigma_a = 693.52$	0.975	0.956	0.938	0.918	0.905
	$\sigma_f = 582.78$	0.975	0.956	0.938	0.915	0.898
^{238}U	$\sigma_a = 2.71$	1.002	1.004	1.006	1.011	1.018
^{239}Pu	$\sigma_a = 1031.10$	1.074	1.163	1.340	1.891	2.829
	$\sigma_f = 747.73$	1.050	1.114	1.246	1.668	2.401
^{240}Pu	$\sigma_a = 300.0$	1.028	1.051	1.082	1.153	1.296

Table III: Neutron Regeneration Data.

	U^{233}	U^{235}	Pu^{239}	Natural Uranium
ν	2.51	2.47	2.90	2.47
η	2.28	2.07	2.10	1.33
$\alpha = \dfrac{\sigma_c}{\sigma_f}$	0.102	0.192	0.38	0.85

EFFECT OF HETEROGENEITY ON THE FUEL UTILIZATION FACTOR

The fuel utilization factor is defined as the ratio of the number of neutrons absorbed per unit time in the fuel, to the total number of neutrons absorbed per unit time in the whole reactor, including fuel, moderator, structure and control elements. For a lumped system consisting of a fuel region and a moderator region:

$$f = \frac{\Sigma_{a1} \int\limits_{fuel} \phi(r)dV}{\Sigma_{a1} \int\limits_{fuel} \phi(r)dV + \Sigma_{a2} \int\limits_{moderator} \phi(r)dV} \qquad (7)$$

Let:

$$\overline{\phi_1} = \frac{\int\limits_{fuel} \phi(r)dV}{V_1}$$

$$\overline{\phi_2} = \frac{\int\limits_{moderator} \phi(r)dV}{V_2}$$

where: V_1, V_2 are the volumes of the fuel and moderator regions, respectively. Then we can write:

$$f = \frac{V_1 \Sigma_{a1} \overline{\phi_1}}{V_1 \Sigma_{a1} \overline{\phi_1} + V_2 \Sigma_{a2} \overline{\phi_2}} \qquad (8)$$

or:

$$f = \frac{1}{1 + \dfrac{\Sigma_{a2}}{\Sigma_{a1}} \dfrac{V_2}{V_1} \dfrac{\overline{\phi_2}}{\overline{\phi_1}}} \qquad (9)$$

The factor $\overline{\phi_2}/\overline{\phi_1}$ is called the "disadvantage factor" and is given by rearranging Eqn. 9 as:

$$\frac{\overline{\phi_2}}{\overline{\phi_1}} = \frac{\Sigma_{a1}}{\Sigma_{a2}} \frac{V_1}{V_2} (\frac{1}{f} - 1) \qquad (10)$$

It is a disadvantage since normally, for a lumped system, $\overline{\phi_2} > \overline{\phi_1}$, consequently $\overline{\phi_2}/\overline{\phi_1} > 1$, and f for a heterogeneous system will be less than for the equivalent homogeneous one as shown in Fig. 1.
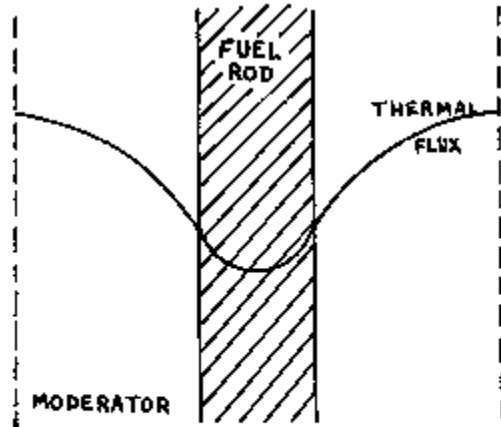

Fig. 1. Thermal Flux Distribution in Lumped
System Unit Cell.

For a lattice configuration, the first step for the calculation of f is to choose a unit cell as shown in Fig.2.
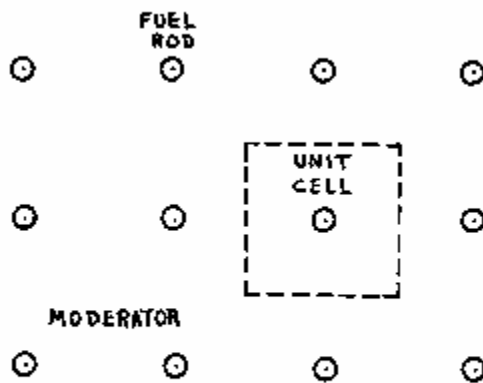

Fig. 2. Unit Cell in a Cylindrical Rods
Lattice Configuration.

The next step is to approximate the true shape of the cell by a shape which can be described by a single dimension, in such a way as to maintain the true volume of the cell.
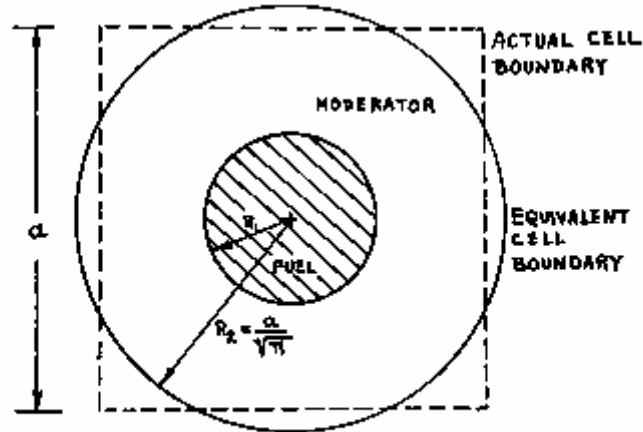

Fig. 3. Cylindrical Equivalent of a Square Unit Cell.

For the cylinder and the square parallelepiped to have the same volume:

$$\pi R^2 L = a^2 L$$

from which:

$$R = \frac{a}{\sqrt{\pi}}$$

With the problem reduced to one dimension, diffusion theory may now be applied. The slowing down density (q) is assumed uniform over the region of the cell occupied by the moderator, and to be zero in the fuel. The diffusion balance equations for this case will be in the fuel and in the moderator:

$$D_1 \nabla^2 \phi_1 - \Sigma_{a1} \phi_1 = 0$$
$$D_2 \nabla^2 \phi_2 - \Sigma_{a2} \phi_2 + q = 0$$

(11)

where the subscript 1 denotes the fuel, and subscript 2 denotes the moderator.

Dividing by the diffusion coefficients, writing: $X^2 = \Sigma_a / D$, and using a cylindrical coordinates system for the Laplacian operator, Eqn.11 becomes:

$$
\frac{d^2\phi_1}{dr^2} + \frac{1}{r}\frac{d\phi_1}{dr} - X_1^2\phi_1 = 0
$$
$$
\frac{d^2\phi_2}{dr^2} + \frac{1}{r}\frac{d\phi_2}{dr} - X_2^2\phi_2 + \frac{q}{D_2} = 0
$$

(12)

The solution in the fuel region in terms of Bessel functions is:

$$
\phi_1 = AI_0(X_1 r) + DK_0(X_1 r) = AI_0(X_1 r)
$$

(13)

since: $D = 0$, as K_0 goes to infinity at $r = 0$, which would lead to an unphysical solution..

The solution in the moderator region is obtained by adding a complementary function and a particular solution:

$$
\phi_2 = BI_0(X_2 r) + CK_0(X_2 r) + \frac{q}{\Sigma_{a2}}
$$

(14)

Since all the cells are identical and are assumed to constitute an infinite array, there must be no neutron current from cell to cell; thus:

$$
\frac{d\phi_2}{dr}(R_2) = 0
$$

(15)

Applying this condition to Eqn. 14, we get the relation between the constants B and C:

$$\frac{d\phi_2(R_2)}{dr} = BX_2 I_1(X_2 R_2) - CX_2 K_1(X_2 R_2) = 0,$$

where we used the Bessel functions the relationships:

$$I_0'(z) = I_1(z),$$
$$K_0'(z) = -K_1(z),$$

We get for the value of C:

$$C = \frac{I_1(X_2 R_2)}{K_1(X_2 R_2)} B \tag{16}$$

Thus Eqn. 14 can be rewritten as:

$$\phi_2 = G[K_1(X_2 R_2)I_0(X_2 r) + I_1(X_2 R_2)K_0(X_2 r)] + \frac{q}{\Sigma_{a2}} \tag{17}$$

where:

$$G = \frac{B}{K_1(X_2 R_2)}$$

Applying the conditions of the continuity of the flux and current at the interface between the fuel: and moderator, for the flux:

$$\phi_1(R_1) = \phi_2(R_1),$$

and for the current

$$D_1\phi_1'(R_1) = D_2\phi_2'(R_1), \tag{18}$$

yields two equations for the values of the constants A and G (although for the evaluation of f, the value of G is not required):

$$AI_0(X_1R_1) = G[K_1(X_2R_2)I_0(X_2R_1) + I_1(X_2R_2)K_0(X_2R_1)] + \frac{q}{\Sigma_{a2}} \quad (19)$$

$$D_1AX_1I_1(X_1R_1) = D_2GX_2[K_1(X_2R_2)I_1(X_2R_1) - I_1(X_2R_2)K_1(X_2R_1)]$$

From the second equation in 19, we get the value of G:

$$G = A \cdot \frac{X_1D_1}{X_2D_2} \cdot \frac{I_1(X_1R_1)}{K_1(X_2R_2)I_1(X_2R_1) - I_1(X_2R_2)K_1(X_2R_1)}$$

Upon substituting the value of G in the first equation in 19, we get the inverse of A as:

$$\frac{1}{A} = \frac{\Sigma_{a2}}{q} \left[I_0(X_1R_1) - \frac{D_1X_1}{D_2X_2} \cdot \frac{I_1(X_1R_1)[K_1(X_2R_2)I_0(X_2R_1) + I_1(X_2R_2)K_0(X_2R_1)]}{K_1(X_2R_2)I_1(X_2R_1) - I_1(X_2R_2)K_1(X_2R_1)} K \right] (20)$$

We recall now the definition of f:

$$f = \frac{A_f}{A_t}$$

and estimate the reaction rate per unit length of the cell. The absorption rate in fuel is just the neutron current into region1, or:

$$A_f = 2\pi R_1 \cdot J(R_1) = 2\pi R_1 \left[D_1 \frac{d\phi_1(R_1)}{dr} \right] = 2\pi R_1 \cdot A \cdot D_1 X_1 I_1(X_1R_1)$$

The total absorption per unit length of cell is just the total source strength per unit length:

$$A_t = \pi(R_2^2 - R_1^2) \cdot q,$$

thus:

$$\frac{1}{f} = \frac{\pi q(R_2^{\;2} - R_1^{\;2})}{2\pi R_1 D_1 I_1(X_1 R_1) A} = \frac{q X_1(R_2^{\;2} - R_1^{\;2})}{2\Sigma_{a1} R_1 I_1(X_1 R_1)} \cdot \frac{1}{A} \tag{21}$$

Inserting the value of 1/A from Eqn. 20 into Eqn. 21:

$$\frac{1}{f} = \frac{V_2}{V_1} \frac{\Sigma_{a2}}{\Sigma_{a1}} \left[\frac{X_1 R_1}{2} \frac{I_0(X_1 R_1)}{I_1(X_1 R_1)} \right] + X_2 \frac{(R_2^{\;2} - R_1^{\;2})}{2R_1} \frac{I_0(X_2 R_1) K_1(X_2 R_2) + K_0(X_2 R_1) I_1(X_2 R_2)}{I_1(X_2 R_2) K_1(X_2 R_1) - K_1(X_2 R_2) I_1(X_2 R_1)} \tag{22}$$

Thus 1/f consists of two terms one which involves X_1 of the fuel only, and the other X_2 of the moderator only. The general form of Eqn. 22 is:

$$\frac{1}{f} = 1 + \frac{V_2 \Sigma_{a2}}{V_1 \Sigma_{a1}} F + (E - 1) \tag{23}$$

The functions E and F are listed for different geometries in Table IV.

If the fuel is surrounded by a thin layer of nonfissionable material of volume V_3 and absorption cross section Σ_{a3} its effect can be included in the expression for 1/f as:

$$\frac{1}{f} = 1 + \left(\frac{V_2 \Sigma_{a2} + V_3 \Sigma_{a3}}{V_1 \Sigma_{a1}} \right) F + (E - 1) \tag{24}$$

we imply here that the effect of the layer on the flux distribution is negligible.

Table IV. The Functions E,F for Different Geometries

Geometry	F	E
Cylinder	$\dfrac{x_1 R_1}{2} \cdot \dfrac{I_0(x_1 R_1)}{I_1(x_1 R_1)}$	$\dfrac{x_2(R_2^2 - R_1^2)}{2R_1} \left[\dfrac{I_0(x_2 R_1)K_1(x_2 R_2) + K_0(x_2 R_1)I_1(x_2 R_2)}{I_1(x_2 R_2)K_1(x_2 R_1) - K_1(x_2 R_2)I_1(x_2 R_1)} \right]$
Slab	$x_1 R_1 \coth(x_1 R_1)$	$x_2(R_2 - R_1)\coth x_2(R_2 - R_1)$
Sphere	$\dfrac{x_1^2 R_1^2}{3} \cdot \dfrac{\tanh(x_1 R_1)}{x_1 R_1 - \tanh x_1 R_1}$	$\dfrac{x_2^2(R_2^3 - R_1^3)}{3R_1} \left[\dfrac{1 - x_2 R_2 \coth x_2(R_2 - R_1)}{1 - x_2 R_2 R_1 - x_2(R_2 - R_1)\coth x_2(R_2 - R_1)} \right]$

For Slab, R_1 is the half-thickness of fuel, R_2 is the half-thickness of the cell.

EFFECT OF HETEROGENEITY ON THE RESONANCE ESCAPE PROBABILITY

The expression for the resonance escape probability for a homogeneous system has been derived as:

$$p = \exp\left(-\frac{N_a I_{eff}}{\xi \Sigma_s} \right) \tag{25}$$

where: I_{eff} is the effective resonance integral.

In case of heterogeneous systems, the resonance absorption inside the fuel is much less than the absorption at the surface. When the fuel material has low moderating power, the energy regions depleted by resonance absorption at the surface are not replenished by moderation, and this "self-shielding" of the fuel core against resonance absorption is significant. This effect makes possible the construction of a heterogeneous reactor using uranium as a moderator. If a homogeneous system is used, criticality cannot be achieved.

A semi-empirical method is used for lumped systems since a well-known resonance structure does not exist. On theoretical and experimental grounds the rate of resonance absorptions per unit time per unit energy interval for uranium lumps of simple shape is given by:

$$\overline{\phi_1} N_1 V_1 a + \overline{\phi_s} N_1 V_1 b \cdot \frac{S}{M}$$

where: $\overline{\phi_s}$ is the average flux per unit energy interval at the surface of the lump,

S is the surface area of the lump,

a, b are experimentally determined constants,

V_1 is the volume of the lump.

These absorptions would lead to a decrease in the slowing-down density given by a factor:

$$V_2 \frac{dq}{dE} \tag{26}$$

From which we can write:

$$V_2 \frac{dq}{dE} = \overline{\phi_1} N_1 V_1 a + \overline{\phi_s} N_1 V_1 b \cdot \frac{S}{M} \tag{27}$$

Recalling the expression of slowing-down density:

$$q = \overline{\phi_2} \xi_2 \Sigma_{s2} E \tag{28}$$

we can write:

$$\frac{dq}{q} = \frac{\overline{\phi_1} N_1 V_1 a + \overline{\phi_s} N_1 V_1 b \cdot \dfrac{S}{M}}{V_2 \overline{\phi_2} \xi_2 \Sigma_{s2} E} \cdot dE$$

Recalling the definition of resonance escape probability, and integrating between the limits E_0 and E_{th}, we get:

$$p = \frac{q_{th}}{q_0} = \exp\left(-\left[\frac{N_1 V_1}{V_2 \xi_2 \Sigma_{s2}} \int_{E_{th}}^{E_0} \frac{\overline{\phi_1}}{\overline{\phi_2}} a \frac{dE}{E} + \frac{S}{M} \int_{E_{th}}^{E_0} \frac{\overline{\phi_s}}{\overline{\phi_2}} b \frac{dE}{E} \right]\right) \qquad (29)$$

If it is assumed that:

$$\frac{\overline{\phi_1}}{\overline{\phi_2}} \approx \frac{\overline{\phi_s}}{\overline{\phi_2}} \approx cons\tan t ,$$

we get:

$$p = \exp\left(-\left[\frac{N_1 V_1}{V_2 \xi_2 \Sigma_{s2}} \frac{\overline{\phi_1}}{\overline{\phi_2}} \int_{E_{th}}^{E_0} \sigma_{aeff} \frac{dE}{E} \right]\right) \qquad (30)$$

where:

$$I_{eff} = \int_{E_{th}}^{E_0} \sigma_{aeff} \frac{dE}{E} = \int_{E_{th}}^{E_0} a \frac{dE}{E} + \frac{S}{M} \int_{E_{th}}^{E_0} b \frac{dE}{E} = A + \mu \frac{S}{M}$$

The values of A and μ are given here for Natural Uranium and some of its compounds:

Material	A[b]	$\mu[\frac{b \cdot g}{cm^2}]$
U	9.25	24.7
U_3O_8	12.00	20.0
UO_2	11.51	22.1
UF_6	14.60	16.3

It still remains to estimate the ratio: $\overline{\phi_1}/\overline{\phi_2}$. In analogy to the previously derived disadvantage factor for the thermal utilization factor:

$$\text{Disadvantage factor} = \frac{\Sigma_{a1}V_1}{\Sigma_{a2}V_2}(\frac{1}{f}-1),$$

We define a "resonance utilization factor" f_r, and can write:

$$\frac{\overline{\phi_1}}{\overline{\phi_2}} = \frac{\Sigma_{a2}V_2}{\Sigma_{a1}V_1} \cdot \frac{1}{(\frac{1}{f_r}-1)} = \frac{\Sigma_{a2}V_2}{\Sigma_{a1}V_1} \cdot \frac{1}{\frac{V_2\Sigma_{a2}}{V_1\Sigma_{a1}}F+(E-1)},$$

Thus:

$$\frac{\overline{\phi_1}}{\overline{\phi_2}} = \frac{1}{F+\frac{\Sigma_{a1}V_1}{\Sigma_{a2}V_2}\cdot(E-1)} \tag{31}$$

which can be substituted into Eq. 30 to yield:

$$p = \exp\left(-\left[\frac{1}{\frac{V_2\xi_2\Sigma_{s2}F}{N_1V_1I_{eff}}+\frac{\Sigma_{a1}\xi_2\Sigma_{s2}(E-1)}{\Sigma_{a2}N_1I_{eff}}}\right]\right) \tag{32}$$

In Eqn. 32, Σ_{a2} denotes a fictitious absorption cross section which is really a slowing-down cross section expressing the probability that a neutron will be slowed down from the resonances energy range.

As previously derived in the context of Fermi Age Theory where it was referenced as a removal cross section, it is given by:

$$\Sigma_{a2} = \frac{\xi_2 \Sigma_{s2}}{\ln \dfrac{E_0}{E_{th}}} \tag{33}$$

The average absorption cross section in the absorber is:

$$\Sigma_{a1} = \frac{N_1 \int \sigma_{aeff} \dfrac{dE}{E}}{\int \dfrac{dE}{E}} = \frac{N_1 I_{eff}}{\ln \dfrac{E_0}{E_{th}}} \tag{34}$$

From Eqn. 33 and Eqn. 34:

$$\frac{\Sigma_{a1}}{\Sigma_{a2}} = \frac{N_1}{\xi_2 \Sigma_{a2}} \cdot I_{eff} \tag{35}$$

Substituting from Eqn. 35 into Eqn. 32, we finally get:

$$p = \exp\left(-\left[\frac{1}{\dfrac{V_2 \xi_2 \Sigma_{s2} F}{N_1 V_1 I_{eff}} + (E-1)}\right]\right) \tag{36}$$

THE THERMAL DIFFUSION COEFFICIENT

For a homogeneous medium the following relationship for the diffusion coefficient, based on transport theory, can be used:

$$D = \cfrac{1}{3\Sigma_t (1-\overline{\mu_0}) \left[1 - \cfrac{4}{5}\cfrac{\Sigma_a}{\Sigma_t} + \cfrac{\Sigma_a}{\Sigma_t}\cfrac{\overline{\mu_0}}{(1-\overline{\mu_0})} + ... \right]} \tag{37}$$

which reduces, for a weakly absorbing medium to:

$$D = \frac{1}{3\Sigma_s (1-\overline{\mu_0})} = \frac{\lambda_{tr}}{3} \tag{38}$$

For heterogeneous reactors, the ratio of moderator volume to fuel volume is large, so that the diffusion coefficient of the moderator can be used in the calculations. If this is not true, the following expression which accounts for the disadvantage factor can be used:

$$\overline{D} = \frac{1}{3\overline{\Sigma_{tr}}} = \frac{1}{3}\frac{V_2\overline{\phi_2} + V_1\overline{\phi_1}}{V_2\overline{\phi_2}\Sigma_{tr2} + V_1\overline{\phi_1}\Sigma_{tr1}} \tag{39}$$

Recalling the expression for the disadvantage factor:

$$\xi = \frac{\overline{\phi_1}}{\overline{\phi_2}} = \frac{V_2\Sigma_{a2}}{V_1\Sigma_{a1}} \cdot \frac{f}{1-f},$$

where f is the thermal utilization factor, we can write it as:

$$\overline{D} = \frac{1}{3}\frac{1 + \xi\dfrac{V_1}{V_2}}{\Sigma_{tr2} + \xi\dfrac{V_1}{V_2}\Sigma_{tr1}} \tag{40}$$

Table V lists values of the transport mean free path for some materials of interest.

THE THERMAL DIFFUSION AREA

The thermal diffusion area is given by:

$$L^2 = \frac{\overline{D}}{\overline{\Sigma_a}} \tag{41}$$

We have already derived an expression for D_{th}, we can write a similar expression for $\overline{\Sigma_a}$ as:

$$\overline{\Sigma_a} = \frac{V_2 \Sigma_{a2} \overline{\phi_2} + V_1 \Sigma_{a1} \overline{\phi_1}}{V_2 \overline{\phi_2} + V_1 \overline{\phi_1}} = \frac{\Sigma_{a2} V_2 (1 + \frac{V_1}{V_2} \frac{\Sigma_{a1}}{\Sigma_{a2}} \xi)}{V_2 + V_1 \xi} = \Sigma_{a2} \cdot \frac{V_2}{V_2 + V_1 \xi} (1 + \frac{f}{1-f})$$

Table V. Transport Mean-free Path for Some
Materials of Interest

Material	Thermal λ_{tr} [cm]
H_2O	0.48
D_2O (0.16%H_2O)	2.65
Be (ρ=1.85)	1.43
BeO (ρ=2.69)	0.90

Table VI. Thermal Diffusion Length for
Different Materials

Material	L [cm]
H_2O	2.85
D_2O (0.16%H_2O)	116.00
Be (ρ=1.85)	20.80
BeO (ρ=2.69)	29.00
Graphite (ρ=1.60)	54.40
Th (ρ=11.2)	2.70
ThO_2 (ρ=16)	4.10
U (ρ=18.9)	1.55
U_3O_8 (ρ=6)	3.70

If $V_2 >> V_1 \xi$, then we can write:

$$\overline{\Sigma_a} \approx \frac{\Sigma_{a2}}{(1-f)} \tag{42}$$

If the diffusion coefficient is taken as being the one for the moderator, then we get:

$$L^2 = \frac{D_2}{\Sigma_{a2}}(1-f) = L_2^2(1-f) \tag{43}$$

where: L_2^2 is the diffusion area of the moderator. Some experimental values for L are given in Table VI.

THE FAST FISSION EFFECT

Non-thermal fissions can occur in the resonance region (e.g. in U^{233}, U^{235} and Pu^{239}), or in the fast region in materials which have a fission threshold in the MeV range (e.g. in U^{238} and Th^{232}).

For heterogeneous systems the calculation of the fast fission factor must account for two phenomena:

1) The probability that a fast neutron born in the fuel lump makes a collision before escaping.

2) The contribution this neutron makes to the fast effect, as well as the contribution of subsequent generations by cascade effects.

Let us introduce:

$$P_n \equiv \quad \text{[probability that a fast neutron produced in the n-th generation fast}$$
$$\text{fission makes a collision before escaping from the fuel lump]}$$

Thus, $(1 - P_n)$ is the probability of escaping from the fuel lump, and $\dfrac{\Sigma_i}{\Sigma_t} P_n$ is the probability of a reaction i, where:

$$\Sigma_t = \Sigma_\gamma + \Sigma_f + \Sigma_{in} + \Sigma_s$$

For the neutron produced in the first generation of fissions, we get the following number of fast neutrons for each first generation neutron:

$$P_1 \left(\frac{\nu \Sigma_f + \Sigma_s}{\Sigma_t} \right) = P_1 \cdot r$$

where we imply that an elastic-scattering reaction has a negligible effect in reducing the neutron energy. From that first generation, some neutrons are lost by escapes, inelastic collisions, and radiative captures and:

[fraction of first generation neutrons entering slowing-down process]$= (1 - P_1) + \dfrac{\Sigma_{in}}{\Sigma_t} P_1$

For the following generations the following cascading process occurs:

Generation	Available Fast Neutrons	Neutrons in Cascade	Neutrons available for slowing-down process
1	1	rP_1	$1[1-P_1 + \frac{\Sigma_{in}}{\Sigma_t} P_1]$
2	rP_1	$r^2 P_1 P_2$	$rP_1[1-P_2 + \frac{\Sigma_{in}}{\Sigma_t} P_2]$
3	$r^2 P_1 P_2$	$r^3 P_1 P_2 P_3$	$r^2 P_1 P_2[1-P_3 + \frac{\Sigma_{in}}{\Sigma_t} P_3]$
⋮	⋮	⋮	⋮
n	$r^{n-1} P_1 \ldots P_{n-1}$	$r^n P_1 \ldots P_n$	$r^{n-1} P_1 \ldots P_{n-1}[1-P_n + \frac{\Sigma_{in}}{\Sigma_t} P_n]$

By adding up the terms in the lost column, we get:

$$\varepsilon \equiv \left[1 - P_1 + \frac{\Sigma_{in}}{\Sigma_t} P_1\right] + rP_1\left[1 - P_2 + \frac{\Sigma_{in}}{\Sigma_t} P_2\right] + \ldots + r^{n-1} P_1 \ldots P_{n-1}\left[1 - P_n + \frac{\Sigma_{in}}{\Sigma_t} P_n\right] + \ldots \quad (44)$$

If we assume that $P_n = P$ for all generations, we get:

$$\varepsilon = \left(1 - P + \frac{\Sigma_{in}}{\Sigma_t} P\right)\left(1 + rP + r^2 P^2 + \ldots + r^{n-1} P^{n-1} + \ldots\right)$$

$$= \left(1 - P + \frac{\Sigma_{in}}{\Sigma_t} P\right) \cdot \frac{1}{1 - rP}, \forall rP < 1 \tag{45}$$

$$= \left(\frac{1 - P + \frac{\Sigma_{in}}{\Sigma_t} P}{1 - P \cdot \left(\frac{\nu\Sigma_f + \Sigma_s}{\Sigma_t}\right)}\right)$$

Subtracting the denominator from the numerator on both sides of Eqn. 45, we get:

$$\varepsilon - 1 = \frac{+P\left[-1 + \frac{\Sigma_{in}}{\Sigma_t} + \frac{\nu\Sigma_f}{\Sigma_t} + \frac{\Sigma_s}{\Sigma_t}\right]}{1 - P\left(\frac{\nu\Sigma_f + \Sigma_s}{\Sigma_t}\right)}$$

$$\varepsilon = 1 + \frac{P}{1 - P\left(\frac{\nu\Sigma_f + \Sigma_s}{\Sigma_t}\right)} \cdot \frac{1}{\Sigma_t}\left(\Sigma_s + \nu\Sigma_f + \Sigma_{in} - \Sigma_t\right)$$

Substituting for Σ_t from Eqn. 43, we get:

$$\varepsilon = 1 + \frac{1}{1 - P\left(\frac{\nu\Sigma_f + \Sigma_s}{\Sigma_t}\right)} \cdot \frac{1}{\Sigma_t}\left(\Sigma_s + \nu\Sigma_f - \Sigma_{in} - \Sigma_\gamma - \Sigma_f + \Sigma_{in} - \Sigma_s\right)$$

And finally:

$$\varepsilon = 1 + \frac{P}{1 - P\left(\frac{\nu\Sigma_f + \Sigma_s}{\Sigma_t}\right)} \cdot \frac{\Sigma_f}{\Sigma_t} \cdot \left(\nu - 1 - \frac{\Sigma_\gamma}{\Sigma_f}\right) \tag{46}$$

We now need to calculate the probability of a collision P. Consider a fuel rod of radius R as shown in Fig. 4. We have already deduced:

(1-P)=Number of neutrons escaping from lump / Number of neutrons produced in the fuel lump
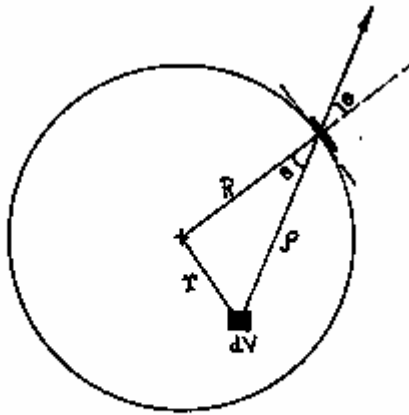

Fig. 4. Fuel Rod Cross section

The denominator is simply:

$$\int_V \nu \Sigma_f \phi(r) dV$$

Writing down the numerator, we get:

$$P = 1 - \frac{\int_S \int_V \nu \Sigma_f \phi(r) \cdot e^{-\Sigma_t \rho} \frac{\cos \theta}{4\pi \rho^2} dV \cdot dS}{\int_V \nu \Sigma_f \phi(r) dV} \qquad (47)$$

where we are considering the number of neutrons produced in volume element dV, attenuated by the factor $e^{-\Sigma_t \rho}$, and by the inverse square distance factor $\dfrac{1}{4\pi\rho^2}$. We are correcting for the cosine of the angle between the normal to the unit area dS and the direction of the outgoing neutrons. An integration over the surface and over the volume is then carried out.

The integration of Eqn. 47 can be tedious and depends on the treated geometry. Values of P for different geometries are shown in Fig. 5 where x is the inverse of the diffusion length, and Σ corresponds to the total cross section. If we assume that the flux is constant in the fuel rod, and that the radius of the rod is small compared to the mean free path, then we can write:
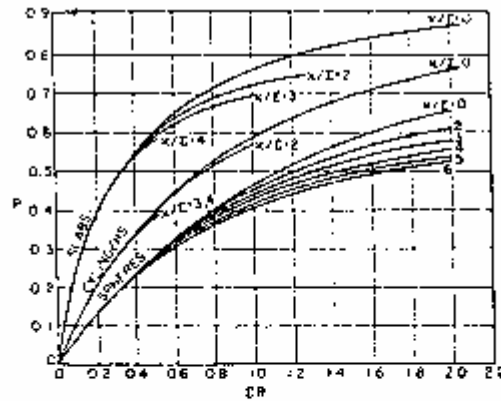
$$P \approx \frac{4}{3}\Sigma_t \cdot R \tag{48}$$

Fig. 5. Dependence of P on ΣR and χ/Σ, where x is the inverse of the diffusion length, Σ is the total cross section, and R is the radius of the fuel pin.

OPTIMIZATION OF LATTICE CONFIGURATION

For a given unit cell pitch, the resonance escape probability decreases as the rod radius is increased, whereas the thermal utilization factor increases to saturation.

For a given rod radius, the resonance escape probability increases with larger cell radius, whereas the thermal utilization factor decreases.
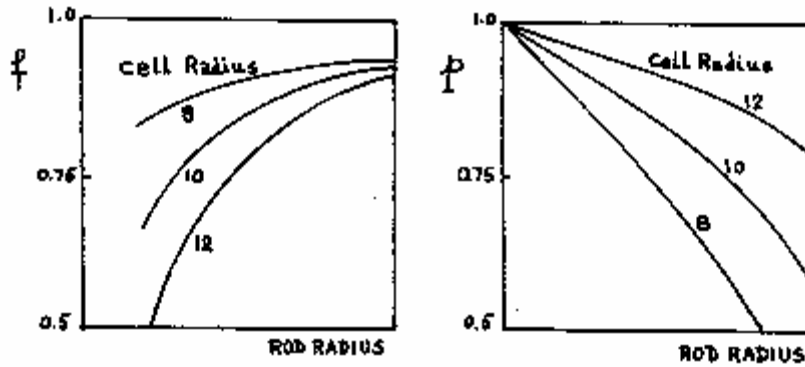

Fig.   Qualitative Behavior of p and f as a
        function of lattice parameters

Thus, it is evident that conditions which favor an increase in f will cause a decrease in p, and vise versa. In the design of a heterogeneous lattice we must then find particular arrangement of a fuel and moderator which gives a maximum value of the four factor formula for the infinite medium multiplication factor:

$$k_\infty = \eta \varepsilon \, p f \qquad\qquad (49)$$

The procedure to be used is thus to consider a number of fuel-moderator lattices, with various rod radii and lattice spacings, and construct a surface for k_∞ as a function of these two parameters. Then choose the lattice that will maximize the value of k_∞. This optimal value occurs normally when the value of p and f are decreasing or increasing at the same rates.

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