MULTIDIMENSIONAL REACTOR SYSTEMS IN DIFFUSION THEORY

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1. INTRODUCTION

The treatment of multidimensional systems, other than the spherical geometry, is an important topic since reactor systems normally have a finite cylinder geometry. Two approaches are possible: numerical methods, and in the simplest case, the separation of variables method can be used for homogeneous systems. We start with a mathematical introduction on orthogonal and orthonormal functions. The method of separation of variables is then used to study the criticality and the flux distribution for the parallelepiped reactor geometry; of which the cube is a special case, and the finite cylinder reactor core which is the geometrical configuration of most existing nuclear power reactors. The treatment will cover the case of multiplying media. The minimum volume for a critical assembly and the peak to average flux ratio will be derived.

2. ORTHOGONAL AND ORTHONORMAL FUNCTIONS

Two functions $\varphi_m(x)$ and $\varphi_n(x)$ are said to be orthogonal over an interval [a,b] if the integral of the product $\varphi_m \varphi_n$ over that interval vanishes:

$$\int_a^b \varphi_m(x)\varphi_n(x)dx = 0$$

In a more general sense, the functions $\varphi_m(x)$ and $\varphi_n(x)$ are said to be orthogonal with respect to a weighting function r(x), over an interval [a, b], if:

$$\int_a^b r(x)\varphi_m(x)\varphi_n(x)dx = 0$$

A set of functions is said to be orthogonal in [a, b] if all pairs of distinct functions in the set are orthogonal in [a, b]

As an application, the one dimensional problem:

$$\frac{d^2X}{dx^2} = -\alpha^2 X$$

with the boundary conditions: X(0) = X(a) = 0, has the eigenvalues or characteristic values:

$$\alpha_n^2 = \left(\frac{n\pi}{a}\right)^2$$

with corresponding characteristic functions or eigenfunctions:

$$X_n(x) = \sin(\frac{n\pi x}{a})$$

Since in this case r(x) = 1, there follows:

$$\int_0^a X_m(x) X_n(x) dx = \int_0^a \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi x}{a}) dx = 0 \text{ , for } m \neq n$$

when m and n are positive integers.

The weighted integral of the square of a characteristic function $\varphi_n(x)$:

$$C_n = \int_a^b r(x) \left[\varphi_n(x)\right]^2 dx$$

has a positive numerical value.

If the arbitrary multiplicative factor involved in the definition of $\varphi_n(x)$ is so chosen that this integral has the value unity, the function $\varphi_n(x)$ is said to be normalized with respect to the weighting function r(x).

A set of normalized orthogonal functions is said to be orthonormal.

By direct integration of the previous equation, we get:

$$C_n = \int_0^a X_n^2 dx = \int_0^a \sin^2(\frac{n\pi x}{a}) = \frac{a}{2}$$

Thus in order to normalize the functions $\sin(\frac{n\pi x}{a})$ over the interval [0, a], we would divide them by the normalizing factor:

$$\sqrt{\frac{a}{2}}.$$

The set of functions:

$$X_n(x) = \sqrt{\frac{2}{a}} \sin(\frac{n\pi x}{a})$$

is thus an orthonormal set in the interval [0,a].

3. THE UNREFLECTED REACTOR PARALLELEPIPED CORE

This is the simplest possible model of a reactor, where a reflector is not used. Consider the geometry of Fig. 1, where the coordinate axes are centered at the origin, and the extrapolated dimensions in the x, y, and z directions are 2a', 2b' and 2c' respectively.

The equation to be solved is the eigenvalue equation:

$$\frac{\partial^2\varphi(x,y,z)}{\partial x^2}+\frac{\partial^2\varphi(x,y,z)}{\partial y^2}+\frac{\partial^2\varphi(x,y,z)}{\partial z^2}+B_g^2\varphi(x,y,z)=0 \qquad (1)$$

The boundary conditions are:

$$\varphi(\pm a',y,z)=\varphi(x,\pm b',z)=\varphi(x,y,\pm c')=0 \qquad (2)$$

We can use the separation of variables method to solve the partial differential equation, by assuming:

$$\varphi(x,y,z)=X(x)Y(y)Z(z) \qquad (3)$$

Substitution into Eq.1 yields:

$$YZ\frac{\partial^2 X}{\partial x^2}+XZ\frac{\partial^2 Y}{\partial y^2}+XY\frac{\partial^2 Z}{\partial z^2}+B_g^2 XYZ=0 \qquad (4)$$

We should replace the partial derivatives by total derivatives. Dividing by XYZ, yields:

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=-B_g^2 \qquad (5)$$

Each term must be separately equal to a constant if it is to hold for all allowed values of x, y, z. This results in three ordinary rather than partial differential equations:

$$\begin{aligned}\frac{1}{X}\frac{d^2X}{dx^2}&=-\alpha^2\\ \frac{1}{Y}\frac{d^2Y}{dy^2}&=-\beta^2\\ \frac{1}{Z}\frac{d^2Z}{dz^2}&=-\gamma^2\end{aligned} \qquad (6)$$

with the condition:

$$\alpha^2 + \beta^2 + \gamma^2 = B_g^2 \tag{7}$$

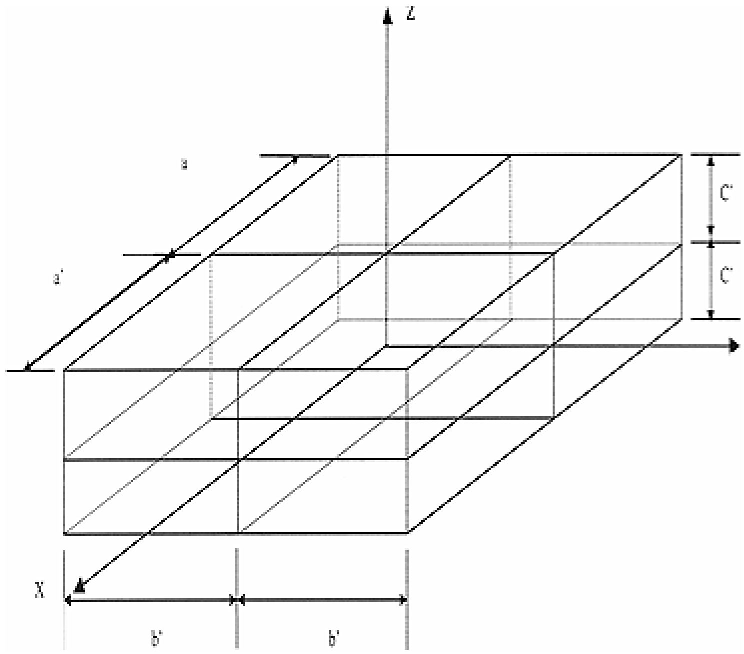

Figure 1. Unreflected or bare parallelepiped reactor core.

Considering the first ordinary second order differential equation in x:

$$\frac{d^2X(x)}{dx^2} = -\alpha^2 X(x) \tag{8}$$

it is known to have the solution:

$$X(x) = A\cos\alpha x + C\sin\alpha x$$

The boundary condition: $X(\pm a') = 0$ requires:

$$A\cos\alpha a' \pm C\sin\alpha a' = 0$$

since we are not interested in the trivial solution A = C = 0, this can be satisfied if:

$$\text{C} = 0;\ \alpha a' = \frac{n\pi}{2} \text{ for all n odd}$$

or by:

$$A = 0;\ \alpha a' = \frac{n\pi}{2}\ \text{ for all n even}$$

Thus:

$$\alpha^2 = \left(\frac{n\pi}{2a'}\right)^2 \tag{9}$$

The even and odd solutions can be taken as members of the normalized set:

$$X_n(x) = \frac{1}{\sqrt{a'}}\cos(\frac{n\pi x}{2a'})\ \text{, for all n odd}$$

$$= \frac{1}{\sqrt{a'}}\sin(\frac{n\pi x}{2a'})\ \text{, for all n even} \tag{10}$$

Similarly for the y and z cases:

$$Y_p(y) = \frac{1}{\sqrt{b'}}\cos(\frac{p\pi y}{2b'})\ \text{, for all p odd}$$

$$= \frac{1}{\sqrt{b'}}\sin(\frac{p\pi y}{2b'})\ \text{, for all p even} \tag{11}$$

Thus:

$$\beta^2 = \left(\frac{p\pi}{2b'}\right)^2 \tag{12}$$

$$Z_q(z) = \frac{1}{\sqrt{c'}}\cos(\frac{q\pi z}{2c'})\ \text{, for all q odd}$$

$$= \frac{1}{\sqrt{c'}}\sin(\frac{q\pi y}{2c'})\ \text{, for all q even} \tag{13}$$

Thus:

$$\gamma^2 = \left(\frac{q\pi}{2c'}\right)^2 \tag{14}$$

The geometrical buckling is the sum of the $B^2_{g(npq)}$ given by Eqns. 7, 9, 12 and 14:

$$B_{g(npq)}^2 = (\frac{n\pi}{2a'})^2 + (\frac{p\pi}{2b'})^2 + (\frac{q\pi}{2c'})^2 \quad (15)$$

The corresponding solution is:

$$\varphi_{npq}(x,y,z) = X_n(x)Y_p(y)Z_q(z) \quad (16)$$

The only choice of n, p, and q which gives a nonnegative flux over the whole core is:

$$n = p = q = 1$$

Thus for a solution we use:

$$B_g{}^2 = B_{g(1,1,1)}^2 = (\frac{\pi}{2a'})^2 + (\frac{\pi}{2b'})^2 + (\frac{\pi}{2c'})^2 \quad (17)$$

since $B_g{}^2$ is fixed by the medium.

Thus there are many choices of the dimensions of the medium to reach criticality, but these dimensions much satisfy the condition 17.

The solution for the critical system becomes from Eqs. 16, 10, 11 and 13:

$$\varphi(x,y,z) = A'\varphi_{111}(x) = A'\frac{1}{\sqrt{a'b'c'}}\cos(\frac{\pi x}{2a'})\cos(\frac{\pi y}{2b'})\cos(\frac{\pi z}{2c'}) \quad (18)$$

4. THE MINIMUM VOLUME OF THE CRITICAL PARALLELEPIPED

Let us minimize the volume of the parallelepiped:

$$V = 8.abc \quad (19)$$

subject to the condition (17).

To introduce the constraint, let us solve for one of the dimensions in terms of B_g and the other two dimensions:

$$a' = a + d = \left[(\frac{2B_g}{\pi})^2 - \frac{1}{b'^2} - \frac{1}{c'^2}\right]^{-\frac{1}{2}} \quad (20)$$

where: d is the extrapolation distance.

On substitution for ‘a’ into the expression for V we get:

$$V = 8\left\{\left[(\frac{2B_g}{\pi})^2 - \frac{1}{(b+d)^2} - \frac{1}{(c+d)^2}\right]^{-\frac{1}{2}} - d\right\} \cdot bc \qquad (21)$$

The minimization proceeds by setting:

$$\frac{\partial V}{\partial b} = \frac{\partial V}{\partial c} = 0$$

This results in two equations in the two unknowns b and c:

$$\frac{\partial V}{\partial b} = 8c\left\{\left[(\frac{2B_g}{\pi})^2 - \frac{1}{(b+d)^2} - \frac{1}{(c+d)^2}\right]^{-\frac{1}{2}} - d\right\} - \frac{8cb}{(b+d)^3} \cdot \frac{1}{2} \cdot 2$$

$$\cdot\left[\left(\frac{2B_g}{\pi}\right)^2 - \frac{1}{(b+d)^2} - \frac{1}{(c+d)^2}\right]^{-\frac{3}{2}} = 0$$

$$\frac{\partial V}{\partial c} = 8b\left\{\left[(\frac{2B_g}{\pi})^2 - \frac{1}{(b+d)^2} - \frac{1}{(c+d)^2}\right]^{-\frac{1}{2}} - d\right\} - \frac{8cb}{(c+d)^3} \cdot \frac{1}{2} \cdot 2$$

$$\cdot\left[\left(\frac{2B_g}{\pi}\right)^2 - \frac{1}{(b+d)^2} - \frac{1}{(c+d)^2}\right]^{-\frac{3}{2}} = 0$$

These two equations yield by using Eqn. 20:

$$a = \frac{b}{(b+d)^3}(a+d)^3$$

$$a = \frac{c}{(c+d)^3}(a+d)^3 \qquad (22)$$

On equating these two expressions for a:

$$\frac{b}{(b+d)^3} = \frac{c}{(c+d)^3} \qquad (23)$$

This implies that:

$$b = c$$

If we would have started by eliminating b instead of a, we would have obtained a = c, thus:

$$a = b = c$$

Thus the critical parallelepiped with minimum volume is found to be a cube with:

$$a' = b' = c' = \sqrt{3}\frac{\pi}{2B_g} \tag{24}$$

5. THE PEAK TO AVERAGE FLUX RATIO

This is an important quantity for heat transfer and fuel management design considerations. This ratio should be as small as possible in order to make the heat generation and the fuel burnup as uniform as possible. Otherwise, larger cooling ducts or orificing must be used in the central parts of the reactor core, and shorter refueling and fuel shuffling times will ensue.

The average flux is given by:

$$\overline{\varphi} = \frac{A}{V}\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}\cos(\frac{\pi x}{2a'})\cos(\frac{\pi y}{2b'})\cos(\frac{\pi z}{2c'})dxdydz$$

$$= A\cdot(\frac{2}{\pi})^3\frac{a'b'c'}{8abc}\left[\sin(\frac{\pi x}{2a'})\right]_{-a}^{a}\left[\sin(\frac{\pi y}{2b'})\right]_{-b}^{b}\left[\sin(\frac{\pi z}{2c'})\right]_{-c}^{c}$$

$$\overline{\varphi} = A\cdot(\frac{4}{\pi})^3\frac{a'b'c'}{8abc}\left[\sin(\frac{\pi x}{2a'})\right]\left[\sin(\frac{\pi y}{2b'})\right]\left[\sin(\frac{\pi z}{2c'})\right] \tag{25}$$

The maximum flux is:

$$\varphi_{\max} = \varphi(x = y = z = 0) = A \tag{26}$$

Thus:

$$\frac{\varphi_{\max}}{\overline{\varphi}} = (\frac{\pi}{2})^3\frac{abc}{a'b'c'}\left[\sin(\frac{\pi a}{2a'})\sin(\frac{\pi b}{2b'})\sin(\frac{\pi c}{2c'})\right]^{-1} \tag{27}$$

If d is small, thus $a' \approx a$, $b' \approx b$, $c' \approx c$ and Eqn. 27 becomes

$$\frac{\varphi_{max}}{\overline{\varphi}} = (\frac{\pi}{2})^3 \qquad (28)$$

a quantity that is independent of the values of a, b, and c.

6. THE FINITE HEIGHT CYLINDRICAL CORE

This is the geometry adopted by most reactor nuclear power plants. In this case the equation to be solved is:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \varphi(r,z)}{\partial r}) + \frac{\partial^2 \varphi(r,z)}{\partial z^2} + B_g^2 \varphi(r,z) = 0 \qquad (29)$$

Assuming a separable solution of the form:

$$\varphi(r,z) = R(r)Z(z) \qquad (30)$$

We get:

$$\frac{1}{R}\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r}) + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + B_g^2 = 0 \qquad (31)$$

Each term must be a constant, thus:

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}(r\frac{dR}{dr}) = -\gamma^2 \qquad (32)$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -\delta^2 \qquad (33)$$

With the constraint:

$$\gamma^2 + \delta^2 = B_g^2 \qquad (34)$$

The equation for Z(z), (Eqn. 33), has a solution:

$$Z(z) = A\cos(\delta z) + C\sin(\delta z) \qquad (35)$$

At the extrapolated height of the cylinder,

$$Z(\pm\frac{h'}{2}) = A\cos(\frac{\delta h'}{2}) \pm C\sin(\frac{\delta h'}{2}) = 0 \qquad (36)$$

Because of symmetry around z = 0, the terms with C are ruled out, thus C = 0, and

$$Z(z) = A\cos(\delta z) \tag{37}$$

and:

$$\frac{\delta h'}{2} = \frac{n\pi}{2} \text{ , for all n = 1, 3, 5…}$$

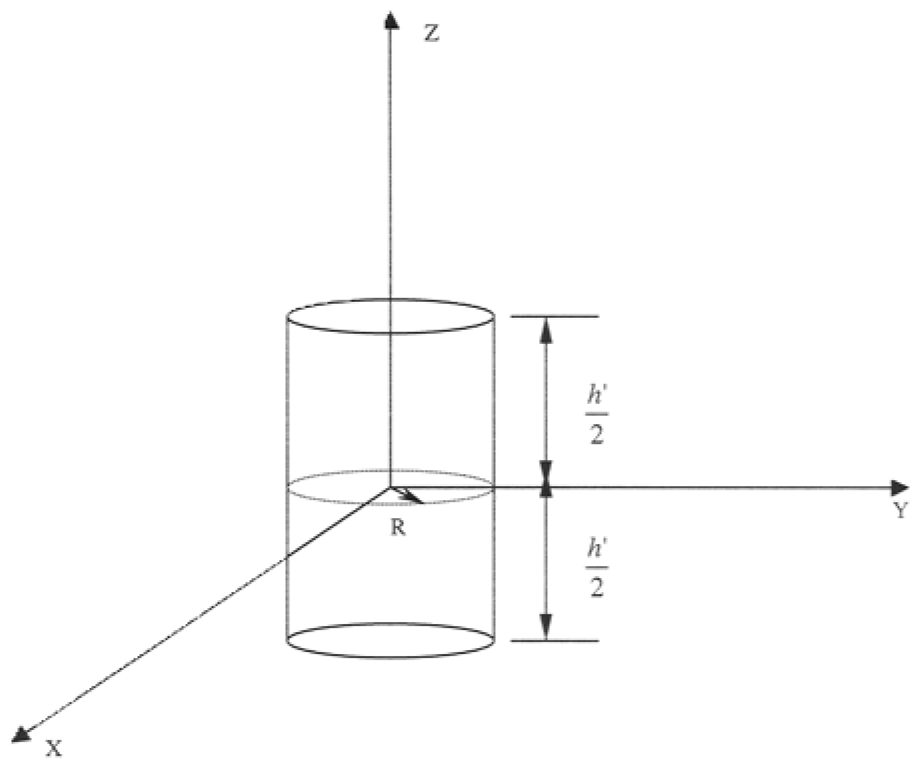

Figure 2. The unreflected finite height cylindrical reactor core.

or:

$$\delta^2 = (\frac{n\pi}{h'})^2 \text{ for all n odd} \tag{38}$$

The equation for R is:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} = -\gamma^2 R \tag{39}$$

which is a Bessel Equation of order zero.

This equation derives its name from the German mathematician and astronomer Frederich Bessel (1784-1846) who reported it while studying planetary motions. In modern engineering practice, it is encountered whenever cylindrical geometry arises in engineering analysis.

The general form of the Bessel Equation of order n, which is a variable coefficient equation is:

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

where n is a constant.

This has a general solution:

$$y(x) = EJ_n(x) + FY_n(x)$$

where: E, F are constants of integration to be determined by the boundary conditions,

$J_n(x)$ is the Bessel function of the first kind of order n,

$Y_n(x)$ is the Bessel function of the second kind of order n, also designated as the Neumann function.

If x is replaced by jx where j = $\sqrt{-1}$, Bessel's Equation modifies into the form:

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 + n^2)y = 0$$

This in turn has a general solution:

$$y(x) = E'I_n(x) + F'K_n(x)$$

where: E', F' are constants of integration to be determined by the boundary conditions,

$I_n(x)$ is the modified Bessel function of the first kind of order n,

$K_n(x)$ is the modified Bessel function of the second kind of order n.

The four Bessel functions of zero order are shown in Fig. 3, and are compared to the cos (x) function.

It can be noticed that both $J_0(x)$ and $Y_0(x)$ are oscillatory. The distance between the roots, or the values at which the functions have a value of zero when they cross the x-axis, become larger and approach the value of π as x increases. The amplitudes of these two functions decrease as x increases, and they are bounded and not infinite everywhere except for $Y_0(x)$ at x = 0, which reaches - ∞.

It is of interest to note that the first root or zero of the function J_0 occurs at x = 2.405.

The two functions, $I_0(x)$ and $K_0(x)$ are non-oscillatory and unbounded, the former going to ∞ at x = ∞, and the latter at x = 0.

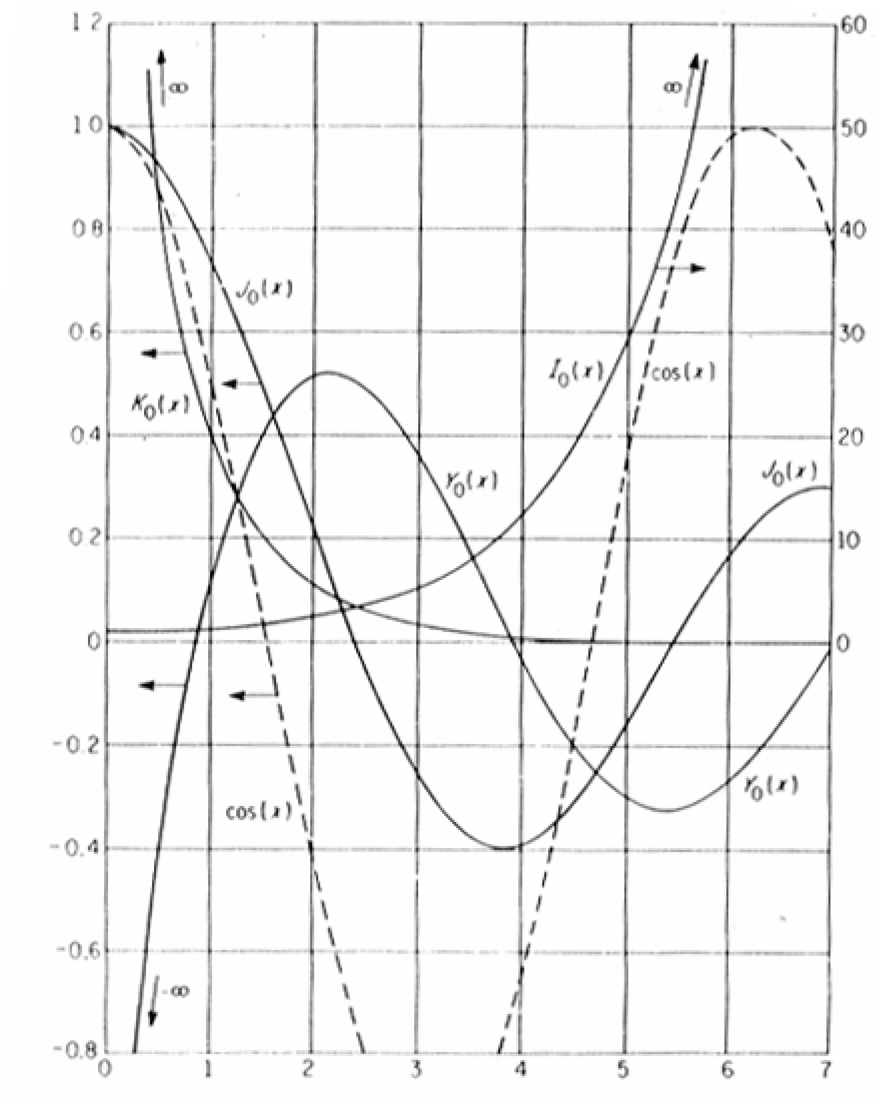

Figure 3. Bessel functions of zero order compared with the cosine function.

The general solution of Eqn. 39 in terms of the Bessel functions of the first and second kind of zero-th order is:

$$R(r) = EJ_0(\gamma r) + FY_0(\gamma r) \quad (40)$$

For the flux to be finite, F = 0, and:

$$R(r) = EJ_0(\gamma r) \quad (41)$$

For the flux to vanish at the extrapolated boundary, $EJ_0(\gamma a') = 0$, we must have:

$$\gamma_p a' = j_{o,p} \text{ , for all p = 1, 2, 3, …}$$

where: $j_{o,p}$ is the argument at which J_0 becomes zero at the p-th time.

For the flux to remain positive we only take p = 1, and $j_{o,p} = 2.405$. Thus:

$$\gamma = \frac{2.405}{a'}$$

and:

$$R(r) = EJ_0(\frac{2.405r}{a'}) \tag{42}$$

Choosing also n = 1 in Eq.38, we get the solution:

$$\varphi(r,z) = \varphi_{\max} J_0(\frac{2.405r}{a'})\cos(\frac{\pi z}{h'}) \tag{43}$$

where: $\varphi_{\max}$ is the flux at the origin.

The criticality condition is:

$$B^2 = \delta_1^2 + \gamma_1^2 = (\frac{\pi}{h'})^2 + (\frac{2.405}{a'})^2 \tag{44}$$

The average flux is given by:

$$\overline{\varphi} = \frac{\varphi_{\max}}{\pi a^2 h}\int_0^a \int_{-\frac{h}{2}}^{\frac{h}{2}} \cos(\frac{\pi z}{h'}) J_o(\frac{2.405r}{a'}) \cdot 2\pi r dr dz$$

$$= \frac{\varphi_{\max}}{\pi a^2 h}[\frac{h'}{\pi} 2\sin(\frac{\pi h}{2h'})] \cdot [2\pi(\frac{a'}{2.405})^2 \frac{2.405a}{a'} J_1(\frac{2.405a}{a'})]$$

where: $\int rJ_0(r)dr = rJ_1(r)$

Thus:

$$\frac{\varphi_{\max}}{\overline{\varphi}} = [\frac{4}{2.405\pi}\frac{a'}{a}\frac{h'}{\pi} 2\sin(\frac{\pi h}{2h'}) J_1(\frac{2.405a}{a'})]^{-1} \tag{45}$$

and for $a' \approx a$, $h' \approx h$, it becomes:

$$\frac{\varphi_{max}}{\overline{\varphi}} = \frac{2.405\pi}{4J_1(2.405)} \approx 1.15\pi \quad (46)$$

Table 1. Geometrical Buckling and flux distribution in different nuclear reactor core geometries.

Reactor core shape	Geometric buckling	Flux distribution
Sphere Radius: R	$B_g^2 = \left(\frac{\pi}{R}\right)^2$	$\phi(r) = A\frac{\sin\left(\frac{\pi r}{R}\right)}{\left(\frac{\pi r}{R}\right)}$
Rectangular parallelepiped Side lengths: a, b, c	$B_g^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2$	$\phi(x,y,z) = A\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)\cos\left(\frac{\pi z}{c}\right)$
Cube Side length: a	$B_g^2 = 3\left(\frac{\pi}{a}\right)^2$	$\phi(x,y,z) = A\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\cos\left(\frac{\pi z}{a}\right)$
Finite height cylinder Radius: R, height: H	$B_g^2 = \left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2$	$\phi(r,z) = AJ_0\left(\frac{2.405r}{R}\right)\cos\left(\frac{\pi z}{H}\right)$
Semi infinite cylinder Radius: R	$B_g^2 = \left(\frac{2.405}{R}\right)^2$	$\phi(r) = AJ_0\left(\frac{2.405r}{R}\right)$
Semi infinite slab Thickness: a	$B_g^2 = \left(\frac{\pi}{a}\right)^2$	$\phi(x) = A\cos\left(\frac{\pi x}{a}\right)$

7. CRITICALITY OF CYLINDRICAL PRESSURIZED WATER REACTOR (PWR) CORE

We consider a bare homogenous cylindrical core with material composition typical of a modern Pressurized Water Reactor (PWR) operating at full power conditions.

The reactor contains a concentration of 2.21 ppb of natural boron as boric acid dissolved in the coolant water and is fuelled with UO_2 at 2.78 percent enrichment in U^{235}.

Based on thermal design considerations, the core height is fixed at H = 3.7 meters.

The macroscopic cross sections for the materials composing this core are as shown in Table 2.

Table 2. Macroscopic cross sections data for a PWR core.

Element/Isotope	Transport Cross section, [cm^{-1}] Σ_{tr}	Absorption Cross section, [cm^{-1}] Σ_a	Product of average number of neutrons released in fission and Fission Cross section, [neutron . cm^{-1}] $\nu\Sigma_f$
H	1.79×10^{-2}	8.08×10^{-3}	-
O	7.16×10^{-3}	4.90×10^{-6}	-
Zr	2.91×10^{-3}	7.01×10^{-4}	-
Fe	9.46×10^{-4}	3.99×10^{-3}	-
U^{235}	3.08×10^{-4}	9.24×10^{-2}	1.45×10^{-1}
U^{238}	6.95×10^{-3}	1.39×10^{-2}	1.20×10^{-2}
B^{10}	8.77×10^{-6}	3.41×10^{-2}	-

We calculate the parameters that are characteristic of a PWR core using the one-group diffusion theory model.

From the table, the summation of the total macroscopic cross sections for the whole homogenized core are;

$$\Sigma_{tr} = 0.03618\ cm^{-1}$$
$$\Sigma_a = 0.1532\ cm^{-1}$$
$$\nu\Sigma_f = 0.157\ n.cm^{-1}$$

The diffusion coefficient is:

$$D = \frac{1}{3}\lambda_{tr} = \frac{1}{3}\frac{1}{\Sigma_{tr}} = \frac{1}{3\times 0.03618} = 9.213 cm$$

The infinite medium multiplication factor becomes:

$$k_\infty = \eta\varepsilon pf = \frac{\nu\Sigma_f}{\Sigma_{aF}}.1.1.\frac{\Sigma_{aF}}{\Sigma_a} = \frac{\nu\Sigma_f}{\Sigma_a} = \frac{0.157}{0.1532} = 1.0248$$

where we have adopted the values:

$$p \approx \varepsilon \approx 1$$

The diffusion area and the diffusion length are:

$$L^2 = \frac{D}{\Sigma_a} = \frac{9.213}{0.1532} = 60.137 cm^2$$
$$L = \sqrt{60.137} = 7.754 cm$$

The material buckling is given by:

$$B_m^2 = \frac{k_\infty - 1}{L^2} = \frac{1.0248 - 1}{60.137} = 4.124 \times 10^{-4} cm^{-2}$$

The extrapolation distance is:

$$d = 0.71\lambda_{tr} = 0.71\frac{1}{\Sigma_{tr}} = \frac{0.71}{0.03618} = 19.62cm$$

The axial geometrical buckling is thus:

$$B_z^2 = \left(\frac{\pi}{H_e}\right)^2 = \left(\frac{\pi}{H + 2d}\right)^2 = \left(\frac{\pi}{370 + (2 \times 19.62)}\right)^2$$
$$= \left(\frac{\pi}{370 + 39.24}\right)^2 = \left(\frac{\pi}{409.24}\right)^2 = (7.68 \times 10^{-3})^2$$
$$= 5.9 \times 10^{-5} cm^{-2}$$

By equating the geometrical buckling to the material buckling as a condition of criticality we obtain the value of the radial geometrical buckling:

$$B_m^2 = B_g^2 = B_r^2 + B_z^2 \Rightarrow B_r^2 = B_m^2 - B_z^2$$
$$B_r^2 = 4.124 \times 10^{-4} - 5.9 \times 10^{-5} = 3.534 \times 10^{-4} cm^{-2}$$

We can thus deduce the value of the critical extrapolated radius as:

$$B_r^2 = \left(\frac{2.405}{R_e}\right)^2 = 3.534 \times 10^{-4} \Rightarrow R_e = \frac{2.405}{\sqrt{3.534 \times 10^{-4}}} = \frac{240.5}{1.88} = 127.9cm$$

There results that the critical radius of the core is:

$$R_c = R_e - d = 127.9 - 19.62 = 108.28\ cm$$

We can compute the critical core volume as:

$$V_c = \pi R_c^2 H = \pi (108.28)^2 \times 370 = 13.63 \times 10^6 cm^3 = 13.6\ m^3$$

One can also estimate the neutrons leakage fraction from the critical core as:

$$P_L = 1 - \ell = 1 - \frac{1}{1 + L^2 B^2} = 1 - \frac{1}{1 + 60.137 \times 4.124 x 10^{-4}}$$
$$= 1 - \frac{1}{1 + 0.0248} = 1 - \frac{1}{1.0248} = 1 - 0.976$$
$$= 0.024 = 2.4 \textit{ percent}$$

REFERENCES

1. M. Ragheb, "Lecture Notes on Fission Reactors Design Theory," FSL-33, University of Illinois, 1982.
2. J. R. Lamarsh, "Introduction to Nuclear Engineering," Addison-Wesley Publishing Company, 1983.

EXERCISES

1. Consider a cubical bare reactor of edge dimensions a = b = c, as a special case of the rectangular parallelepiped reactor core.
Use separation of variables to solve from first principles for the neutron flux in the reactor as a function of position, and derive an expression for the geometric buckling.

2. Consider a cylindrical bare reactor of unit height to diameter ratio: H = 2R, where H is its height and R is its radius, as a special case of the finite cylindrical reactor core.
Use separation of variables to solve from first principles for the neutron flux in the reactor as a function of position, and derive an expression for the geometric buckling.

3. Choose a different critical core configuration with a unity height to diameter ratio and with the same material compositions, and recalculate the reactor parameters for the typical PWR core. Discuss your results.

4. For:

a) A spherical reactor core with R = 20 cm,

b) A cubical reactor core with a = 40 cm, and

c) A cylindrical reactor core with H = 2R = 40 cm,

containing a mixture of U^{235} as fuel and graphite as a moderator, compare the moderator to fuel ratios : S = Ng / Nu that will achieve criticality for each configuration.

Use: ρ(graphite)=1.6 g/cm^3, microscopic absorption cross-section of graphite $\sigma_a = 3.4x10^{-3}$ b, microscopic absorption cross-section of U^{235}=681 b, ν = 2.07, D = 0.85 cm.

5. Compare the critical masses of fast reactors composed of U^{235} in the following geometrical shapes:
a) A spherical reactor core.
b) A cubical reactor core.
c) A cylindrical reactor core with H = 2R.

Use:
microscopic transport cross section = 8.246 [barns]
microscopic absorption cross section = 2.844 [barns]
density = 18.75 [gm/cm^3]
product of average number of neutrons released in fission (ν) and the microscopic fission cross section = 5.297 [neutrons.barn].

6. Using the one group steady state neutron diffusion equation and ignoring the extrapolation lengths, derive the expression for the flux distribution in a **finite** height cylindrical reactor core of radius R and height H as shown in Fig. 1.
Apply the appropriate boundary conditions and derive the expression for the geometrical buckling for such a reactor core.
By equating the geometrical buckling to the material buckling, derive the one group criticality equation for the finite height cylindrical core.
Generalize the one group criticality equation to a two group formulation including a fast neutrons group with Fermi age τ and a thermal group with diffusion area L^2.
For a large reactor deduce the modified one group criticality equation in terms of the migration area $M^2 = \tau + L^2$.

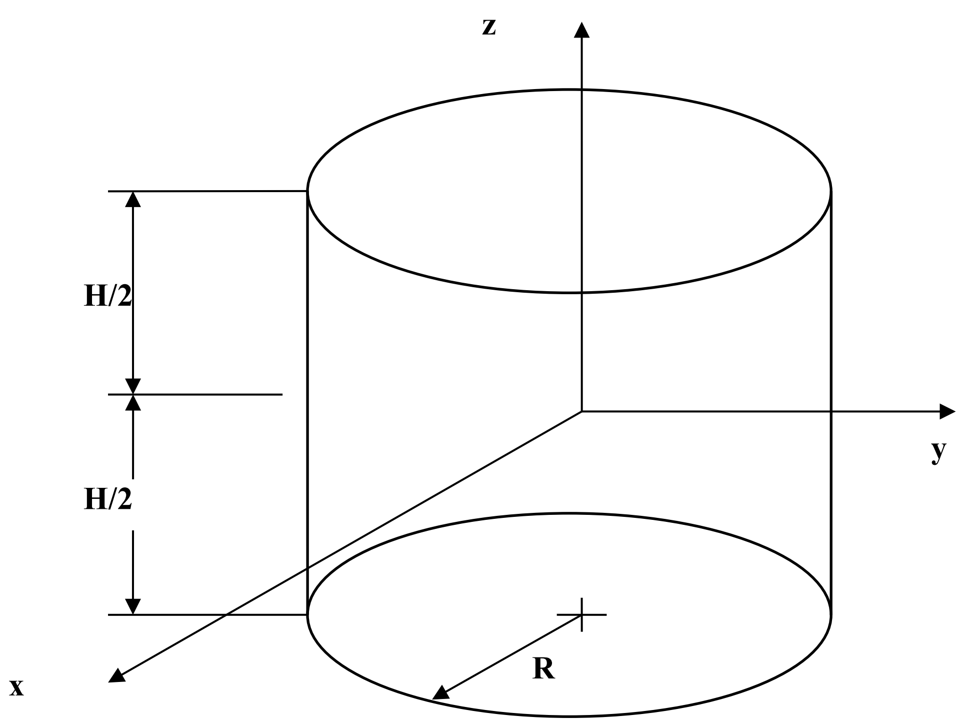

Figure 1. Finite cylindrical core of height H and radius R.

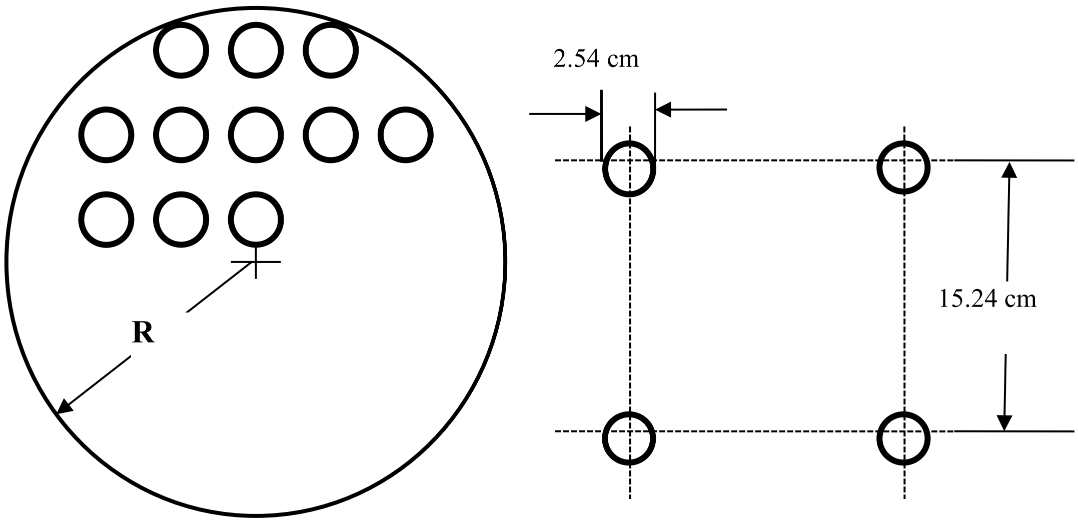

Figure 2. Lattice configuration and the unit cell of the natural uranium rods in the D_2O moderator.

Consider a D_2O cooled and moderated reactor containing 2.54 cm diameter vertical rods of natural uranium arranged in a square lattice configuration with pitch (spacing) of 15.24 cm, and suspended in the D_2O as shown in Fig. 2.
The height to diameter ratio (H/2R) of the cylindrical core is 1.2.
The infinite medium multiplication factor $k_\infty = 1.28$.
The diffusion area for thermal neutrons is $L^2 = 175\ cm^2$, and the Fermi age for fast neutrons is $\tau = 120\ cm^2$.
Calculate the following reactor parameters:
a. Material buckling.
b. Critical radius, critical height and critical volume.
c. Fast neutrons non-leakage probability P_f and thermal neutrons non-leakage probability P_{th}
d. Estimate of the number of natural uranium fuel rods that would fit in this core.
e. Calculate the weight of natural uranium and of the heavy water to be procured to make this reactor just critical and the total weight of the whole reactor core in metric tonnes. The density of the natural uranium metal can be taken as 19 gm/cm^3, and the density of heavy water as 1.1 gm/cm^3.